CURVATURE ESTIMATES FOR THE LEVEL SETS OF SPATIAL QUASICONCAVE SOLUTIONS TO A CLASS OF PARABOLIC EQUATIONS

CHUANQIANG CHEN AND SHUJUN SHI

Abstract. We prove a constant rank theorem for the second fundamental form of the spatial convex level surfaces of solutions to equations $u_t = F(\nabla^2 u, \nabla u, u, t)$ under a structural condition, and give a geometric lower bound of the principal curvature of the spatial level surfaces.

1. Introduction

In this paper, we consider the convexity and principal curvature estimates of the spatial level surfaces of the spatial quasiconcave solutions to a class of parabolic equations under some structural conditions. A continuous function $u(x,t)$ on $\Omega \times [0,T]$ is called *spatial quasiconcave* if its level sets $\{x \in \Omega | u(x,t) \geqslant c\}$ are convex for each constant c and any fixed $t \in [0,T]$.

The convexity of the level sets of the solutions to elliptic partial differential equations has been studied extensively. For instance, Ahlfors [1] contains the well-known result that level curves of Green function on simply connected convex domain in the plane are the convex Jordan curves. In 1956, Shiffman [20] studied the minimal annulus in $\mathbb{R}^3$ whose boundary consists of two closed convex curves in parallel planes P_1, P_2. He proved that the intersection of the surface with any parallel plane P, between P_1 and P_2, is a convex Jordan curve. In 1957, Gabriel [9] proved that the level sets of the Green function on a 3-dimensional bounded convex domain are strictly convex. In 1977, Lewis [14] extended Gabriel's result to p-harmonic functions in higher dimensions. Caffarelli-Spruck [7] generalized the Lewis [14] results to a class of semilinear elliptic partial differential equations. Motivated by the result of Caffarelli-Friedman [6], Korevaar [13] gave a new proof on the results of Gabriel and Lewis by applying the deformation process and the constant rank theorem of the second fundamental form of the convex level sets of p-harmonic function. A survey of this subject is given by Kawohl [12]. For more recent related extensions, please see the papers by Bianchini-Longinetti-Salani [4], Bian-Guan [2], Xu [23] and Bian-Guan-Ma-Xu [3].

2000 Mathematics Subject Classification: 45B99, 35K10.

Keywords and phrases: curvature estimates, level sets, constant rank theorem, spatial quasiconcave solutions.

Research of the first author was supported by Grant 10871187 from the National Natural Science Foundation of China. Research of the second author was supported in part by the Science Research Program from the Education Department of Heilongjiang Province (11551137).

There is also an extensive literature on the curvature estimates of the level sets of the solutions to elliptic partial differential equations. For 2-dimensional harmonic function and minimal surface with convex level curves, Ortel-Schneider [19], Longinetti [15] and [16] proved that the curvature of the level curves attains its minimum on the boundary (see Talenti [21] for related results). Longinetti also studied the precise relation between the curvature of the convex level curves and the height of 2-dimensional minimal surface in [16]. Ma-Ou-Zhang [17] got the Gaussian curvature estimates of the convex level sets on higher dimensional harmonic function, and Wang-Zhang [22] got the similar curvature estimates of some quasi-linear elliptic equations under certain structure condition [4]. Both of their test functions involved the Gaussian curvature of the boundary and the norm of the gradient on the boundary. Furthermore, for the p-harmonic function with strictly convex level sets, Ma-Zhang [18] obtained that the curvature function introduced in it is concave with respect to the height of the p-harmornic function. For the principal curvature estimates in higher dimension, in terms of the principal curvature of the boundary and the norm of the gradient on the boundary, Chang-Ma-Yang [8] obtained the lower bound estimates of principal curvature for the strictly convex level sets of higher dimensional harmonic functions and solutions to a class of semilinear elliptic equations under certain structure condition [4]. Recently, in Guan-Xu [11], they got a lower bound for the principal curvature of the level sets of solutions to a class of fully nonlinear elliptic equations in convex rings under the general structure condition [4] via the approach of constant rank theorem.

Naturally, we hope to give a characterization about the convexity and curvature of the level surfaces of the solutions to the corresponding parabolic equations. Borell [5] showed the same property in [9] and [14] for the solution of the corresponding heat conduction problem with zero initial data. In this paper, we will consider the following parabolic equations

$$\frac{\partial u}{\partial t} = F(\nabla^2 u, \nabla u, u, t), \text{ in } \Omega \times (0, T], \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^n$, and $\nabla^2 u$, ∇u are the spatial Hessian and spatial gradient of $u(x,t)$ respectively. Let $\mathcal{S}^n$ denote the space of real symmetric $n \times n$ matrices, $\Lambda \subset \mathcal{S}^n$ an open set, $\mathbb{S}^{n-1}$ a unit sphere and $F = F(r, p, u, t)$ a $C^{2,1}$ function in $\Lambda \times \mathbb{R}^n \times \mathbb{R} \times [0, T]$. We will assume that F satisfies the following conditions: there are $\gamma_0 > 0$ and $c_0 \in \mathbb{R}$,

$$F^{\alpha\beta} := \left(\frac{\partial F}{\partial r_{\alpha\beta}}(r, p, u, t)\right) > 0, \quad \forall\ (r, p, u, t) \in \Lambda \times \mathbb{R}^n \times (-\gamma_0 + c_0, \gamma_0 + c_0) \times [0, T], \tag{1.2}$$

and for each $(\theta, u) \in \mathbb{S}^{n-1} \times \mathbb{R}$ fixed,

$$F(s^2 A, s\theta, u, t) \text{ is locally concave in } (A, s) \text{ for each fixed } t. \tag{1.3}$$

Now we state our theorems.

Theorem 1.1. *Suppose $u \in C^{3,1}(\Omega \times [0, T])$ is a spatial quasiconcave solution to parabolic equation (1.1) such that $(\nabla^2 u(x,t), \nabla u(x,t), u(x,t)) \in \Lambda \times \mathbb{R}^n \times (-\gamma_0 + c_0, \gamma_0 + c_0)$ for each $(x,t) \in \Omega \times [0, T]$. Suppose that, F satisfies conditions (1.2) and (1.3), $\nabla u \neq 0$ and the spatial level sets $\{x \in \Omega | u(x,t) \geqslant c\}$ of u are connected and locally convex for all*

$c \in (-\gamma_0 + c_0, \gamma_0 + c_0)$ for some $\gamma_0 > 0$. Then the second fundamental form of spatial level surfaces $\{x \in \Omega | u(x,t) = c\}$ has the same constant rank for all $c \in (-\gamma_0 + c_0, \gamma_0 + c_0)$. Moreover, let $l(t)$ be the minimal rank of the second fundamental form in Ω, then $l(s) \leqslant l(t)$ for all $s \leqslant t \leqslant T$.

Inspired by [11], we also consider to establish a geometric lower bound for the principal curvature of the spatial level surfaces of solutions to parabolic equation on the convex rings as follows,

$$
\begin{cases}
\frac{\partial u}{\partial t} = F(\nabla^2 u, \nabla u, u, t) & \text{in} \quad \Omega \times (0,T], \\
u(x,0) = u_0(x) & \text{in} \quad \Omega, \\
u(x,t) = 0 & \text{on} \quad \partial\Omega_0 \times (0,T], \\
u(x,t) = 1 & \text{on} \quad \partial\Omega_1 \times (0,T],
\end{cases}
\tag{1.4}
$$

where $\Omega = \Omega_0 \backslash \overline{\Omega_1}$, Ω_0, Ω_1 are two convex domains with $\overline{\Omega_1} \subset \Omega_0$, $F(\nabla^2 u_0, \nabla u_0, u_0, 0) > 0$ and u_0 is quasiconcave and satisfies

$$
\begin{cases}
u_0 = 0 & \text{on} \quad \partial\Omega_0, \\
u_0 = 1 & \text{on} \quad \partial\Omega_1.
\end{cases}
\tag{1.5}
$$

We denote $\kappa_s(x,t)$ the smallest principal curvature of the spatial level set $\Sigma^{u(x_0,t)} = \{x \in \Omega | u(x,t) = u(x_0,t)\}$ at (x,t). For each (x_0,t), set

$$
\kappa^{u(x_0,t)} = \inf_{x \in \Sigma^{u(x_0,t)}} \kappa_s(x,t).
\tag{1.6}
$$

We will assume that there exists $\lambda > 0$, such that

$$
(F^{\alpha\beta}(\nabla^2 u, \nabla u, u, t)) \geqslant \lambda(\delta_{\alpha\beta}), \quad \forall (x,t) \in \overline{\Omega} \times [0,T].
\tag{1.7}
$$

Theorem 1.2. *Suppose $u \in C^{3,1}(\Omega \times [0,T])$ is a spatial quasiconcave solution to parabolic equation (1.4), and F satisfies conditions (1.7) and (1.3), $\nabla u \neq 0$, then*

$$
\kappa^{u(x,t)} \geqslant \min\{\kappa^0, \kappa^1 e^{-A}\} e^{Au(x,t)}
\tag{1.8}
$$

for some universal constant A depending only on $\|F\|_{C^2}$, n, λ, $\min\limits_{(x,t)\in\overline{\Omega}\times[0,T]} |\nabla u|$, $\|u\|_{C^3}$. Moreover, if " $=$ " holds for some $u(x,t) \in (0,1)$, then the " $=$ " holds for all $u(x,t) \in [0,1]$.

Theorem 1.1 and Theorem 1.2 may be looked as some parabolic versions for Theorem 1.1 in [3] and Theorem 1.5 in [11] respectively. The main idea to prove the main theorems in this paper can be found in the two literatures.

The rest of the paper is organized as follows. In section 2, we prove Theorem 1.1. In section 3, we prove Theorem 1.2.

Acknowledgement The authors would like to express sincere gratitude to Prof. Xi-Nan Ma for his encouragement and many suggestions in this subject.

2. Proof of Theorem 1.1

Suppose $u(x,t) \in C^{3,1}(\Omega \times [0.T])$, and $u_n \neq 0$ for any fixed $(x,t) \in \Omega \times [0,T]$. It follows that the upward inner normal direction of the spatial level sets $\{x \in \Omega | u(x,t) = c\}$ is

$$\vec{n} = \frac{|u_n|}{|\nabla u| u_n}(u_1, u_2, ..., u_{n-1}, u_n), \tag{2.1}$$

where $\nabla u = (u_1, u_2, ..., u_{n-1}, u_n)$ is the spatial gradient of u.

The second fundamental form II of the spatial level surface of function u with respect to the upward normal direction (2.1) is

$$b_{ij} = -\frac{|u_n|(u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn})}{|\nabla u| u_n^3}. \tag{2.2}$$

Set

$$h_{ij} = u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn}, \tag{2.3}$$

we may write

$$b_{ij} = -\frac{|u_n| h_{ij}}{|\nabla u| u_n^3}. \tag{2.4}$$

Note that if $\Sigma^{c,t} = \{x \in \Omega | u(x,t) = c\}$ is locally convex, then the second fundamental form of $\Sigma^{c,t}$ is semipositive definite with respect to the upward normal direction (2.1). Let $a(x,t) = (a_{ij}(x,t))$ be the symmetric Weingarten tensor of $\Sigma^{c,t} = \{x \in \Omega | u(x,t) = c\}$, then a is semipositive definite. As computed in [3], if $u_n \neq 0$, and the Weingarten tensor is

$$a_{ij} = -\frac{|u_n|}{|\nabla u| {u_n}^3} \left\{ h_{ij} - \frac{u_i u_l h_{jl}}{W(1+W)u_n^2} - \frac{u_j u_l h_{il}}{W(1+W)u_n^2} + \frac{u_i u_j u_k u_l h_{kl}}{W^2(1+W)^2 u_n^4} \right\}. \tag{2.5}$$

With the above notations, at the point (x,t) where $u_n(x,t) = |\nabla u(x,t)| > 0$, $u_i(x,t) = 0$, $i = 1, \cdots, n-1$, $a_{ij,k}$ is commutative, that is, they satisfy the Codazzi property $a_{ij,k} = a_{ik,j}$, $\forall i,j,k \leqslant n-1$.

2.1. **Calculations on the test function.** Since Theorem 1.1 is of local feature, we may assume level surface $\Sigma^{c,t} = \{x \in \Omega | u(x,t) = c\}$ is connected for each $c \in (c_0 - \gamma_0, c_0 + \gamma_0)$. Suppose $a(x,t_0)$ attains minimal rank $l = l(t_0)$ at some point $z_0 \in \Omega$. We may assume $l \leqslant n-2$, otherwise there is nothing to prove. And we assume $u \in C^{3,1}(\Omega \times [0,T])$ and $u_n > 0$ in the rest of this paper. So there is a neighborhood $\mathcal{O} \times (t_0 - \delta, t_0 + \delta]$ of (z_0, t_0), such that there are l "good" eigenvalues of (a_{ij}) which are bounded below by a positive constant, and the other $n-1-l$ "bad" eigenvalues of (a_{ij}) are very small. Denote G be the index set of these "good" eigenvalues and B be the index set of "bad" eigenvalues. And for any fixed point $(x,t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta]$, we may express (a_{ij}) in a form of (2.5), by choosing $e_1, \cdots, e_{n-1}, e_n$ such that

$$|\nabla u(x,t)| = u_n(x,t) > 0 \text{ and} (u_{ij}), i,j = 1,..,n-1, \text{is diagonal at } (x,t). \tag{2.6}$$

Without loss of generality we assume $u_{11} \geqslant u_{22} \geqslant \cdots \geqslant u_{n-1n-1}$. So, at $(x,t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta)$, from (2.5), we have the matrix $(a_{ij}), i,j = 1,..,n-1$, is also diagonal,

and without loss of generality we may assume $a_{11} \geqslant a_{22} \geqslant ... \geqslant a_{n-1,n-1}$. There is a positive constant $C > 0$ depending only on $\|u\|_{C^4}$ and $\mathcal{O} \times (t_0 - \delta, t_0 + \delta]$, such that $a_{11} \geqslant a_{22} \geqslant ... \geqslant a_{ll} > C$ for all $(x,t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta)$. For convenience we denote $G = \{1, \cdots, l\}$ and $B = \{l+1, \cdots, n-1\}$ be the "good" and "bad" sets of indices respectively. If there is no confusion, we also denote

$$G = \{a_{11}, ..., a_{ll}\} \textit{ and } B = \{a_{l+1,l+1}, ..., a_{n-1,n-1}\}. \tag{2.7}$$

Note that for any $\delta > 0$, we may choose $\mathcal{O} \times (t_0 - \delta, t_0 + \delta]$ small enough such that $a_{jj} < \delta$ for all $j \in B$ and $(x,t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta]$.

For each c, let $a = (a_{ij})$ be the symmetric Weingarten tensor of $\Sigma^{c,t}$. Set

$$p(a) = \sigma_{l+1}(a_{ij}), \quad q(a) = \begin{cases} \frac{\sigma_{l+2}(a_{ij})}{\sigma_{l+1}(a_{ij})}, & \text{if } \sigma_{l+1}(a_{ij}) > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2.8}$$

Theorem 1.1 is equivalent to say $p(a) \equiv 0$ (defined in (2.8)) in $\mathcal{O} \times (t_0 - \delta, t_0]$. Since we are dealing with general fully nonlinear equation (1.1), as in the case for the convexity of solutions in [2], there are technical difficulties to deal with $p(a)$ alone. A key idea in [2] is the introduction of function q as in (2.8) and explore some crucial concavity properties of q. We consider function

$$\phi(a) = p(a) + q(a), \tag{2.9}$$

where p and q as in (2.8). We will use notion $h = O(f)$ if $|h(x,t)| \leqslant Cf(x,t)$ for $(x,t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta]$ with positive constant C under control.

To get around $p = 0$, for $\varepsilon > 0$ sufficiently small, we instead consider

$$\phi_\varepsilon(a) = \phi(a_\varepsilon), \tag{2.10}$$

where $a_\varepsilon = a + \varepsilon I$. We will also denote $G_\varepsilon = \{a_{ii} + \varepsilon, i \in G\}$, $B_\varepsilon = \{a_{ii} + \varepsilon, i \in B\}$.

To simplify the notations, we will drop subindex ε with the understanding that all the estimates will be independent of ε. In this setting, if we pick $\mathcal{O} \times (t_0 - \delta, t_0 + \delta]$ small enough, there is $C > 0$ independent of ε such that

$$\phi(a(x,t)) \geqslant C\varepsilon, \quad \sigma_1(B) \geqslant C\varepsilon, \quad \text{for all } (x,t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta]. \tag{2.11}$$

In what follows, we will use $i, j, \cdots$ as indices run from 1 to $n-1$ and use the Greek indices $\alpha, \beta, \cdots$ as indices run from 1 to n. Denote

$$\begin{aligned} &F^{\alpha\beta} = \frac{\partial F}{\partial u_{\alpha\beta}}, F^{p_\alpha} = \frac{\partial F}{\partial u_\alpha}, F^u = \frac{\partial F}{\partial u}, F^t = \frac{\partial F}{\partial t}, \\ &F^{\alpha\beta,\gamma\eta} = \frac{\partial^2 F}{\partial u_{\alpha\beta}\partial u_{\gamma\eta}}, F^{\alpha\beta,p_\gamma} = \frac{\partial^2 F}{\partial u_{\alpha\beta}\partial u_\gamma}, F^{\alpha\beta,u} = \frac{\partial^2 F}{\partial u_{\alpha\beta}\partial u}, \\ &F^{p_\alpha p_\beta} = \frac{\partial^2 F}{\partial u_\alpha \partial u_\beta}, F^{p_\alpha,u} = \frac{\partial^2 F}{\partial u_\alpha \partial u}, F^{u,u} = \frac{\partial^2 F}{\partial u^2}. \end{aligned}$$

We also denote

$$\mathcal{H}_\phi = \sum_{i,j \in B} |\nabla a_{ij}| + \phi. \tag{2.12}$$

Lemma 2.1. *For any fixed $(x,t) \in \mathcal{O} \times (t_0-\delta, t_0+\delta]$, with the coordinate chosen as in (2.6) and (2.7),*

$$\phi_t = -u_n^{-3} \sum_{j \in B} \left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right] [u_n^2 u_{jjt} - 2u_n u_{jn} u_{jt}] + O(\mathcal{H}_\phi) \tag{2.13}$$

and

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^n F^{\alpha\beta}\phi_{\alpha\beta} \\
= \ & u_n^{-3} \sum_{j \in B} \left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right] [-u_n^2 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta jj} + 2u_n u_{nj} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta j} \\
& \qquad + 4u_n u_{nj} \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta j} - 6u_{nj}^2 \sum_{\alpha,\beta=1}^n F^{\alpha\beta} u_{\alpha\beta}] \\
& + 2u_n^{-3} \sum_{j \in B, i \in G} \left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right] \sum_{\alpha,\beta=1}^n F^{\alpha\beta} \frac{1}{u_{ii}} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}][u_n u_{ij\beta} - 2u_{i\beta} u_{jn}] \\
& - \frac{1}{\sigma_1^3(B)} \sum_{\alpha,\beta=1}^n \sum_{i \in B} F^{\alpha\beta} [\sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\alpha}][\sigma_1(B) a_{ii,\beta} - a_{ii} \sum_{j \in B} a_{jj,\beta}] \\
& - \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^n \sum_{i \neq j \in B} F^{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} + O(\mathcal{H}_\phi).
\end{aligned}$$

Proof: For any fixed point $(x,t) \in \mathcal{O} \times (t_0-\delta, t_0+\delta]$, choose a coordinate system as in (2.6) so that $|\nabla u| = u_n > 0$ and the matrix $(a_{ij}(x,t))$ is diagonal for $1 \leqslant i,j \leqslant n-1$ and nonnegative. From the definition of ϕ,

$$a_{jj} = -\frac{h_{jj}}{u_n^3} = -\frac{u_{jj}}{u_n} = O(\mathcal{H}_\phi), \forall j \in B, \tag{2.14}$$

and

$$\begin{aligned}
\phi_t &= \sum_{j \in B} \left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right] a_{jj,t} + O(\mathcal{H}_\phi) \\
&= -u_n^{-3} \sum_{j \in B} \left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right] [u_n^2 u_{jj,t} - 2u_n u_{jn} u_{jt}] + O(\mathcal{H}_\phi) \qquad (2.15)
\end{aligned}$$

Using relationship (2.14), we have

$$\begin{aligned}
\phi_{\alpha\beta} \;=\; & \sum_{j\in B}\left[\sigma_l(G)+\frac{\sigma_1^2(B|j)-\sigma_2(B|j)}{\sigma_1^2(B)}\right]\left[a_{jj,\alpha\beta}-2\sum_{i\in G}\frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}}\right]\\
& -\frac{1}{\sigma_1^3(B)}\sum_{i\in B}\left[\sigma_1(B)a_{ii,\alpha}-a_{ii}\sum_{j\in B}a_{jj,\alpha}\right]\left[\sigma_1(B)a_{ii,\beta}-a_{ii}\sum_{j\in B}a_{jj,\beta}\right]\\
& -\frac{1}{\sigma_1(B)}\sum_{i\neq j\in B}a_{ij,\alpha}a_{ij,\beta}+O(\mathcal{H}_\phi).
\end{aligned}\tag{2.16}$$

So far, we have followed standard calculations as in [10, 3, 2]. Since $u_k=0$ for $k=1,\cdots,n-1$, from (2.5),

$$u_nu_{ij\alpha}=-u_n^2a_{ij,\alpha}+u_{nj}u_{i\alpha}+u_{ni}u_{j\alpha}+u_{n\alpha}u_{ij},\quad \forall\, i,j\leqslant n-1,\tag{2.17}$$

and for each $j\in B$,

$$\begin{aligned}
a_{jj,\alpha\beta} \;=\; & -\frac{1}{u_n^3}h_{jj,\alpha\beta}+O(\mathcal{H}_\phi)\\
=\; & -\frac{1}{u_n^3}[u_n^2u_{jj\alpha\beta}+2u_{nn}u_{j\alpha}u_{j\beta}+2u_{n\alpha}u_{nj}u_{j\beta}+2u_{n\beta}u_{nj}u_{j\alpha}\\
& -2u_nu_{nj}u_{\alpha\beta j}-2u_nu_{j\alpha}u_{nj\beta}-2u_nu_{j\beta}u_{nj\alpha}]+O(\mathcal{H}_\phi).
\end{aligned}\tag{2.18}$$

Hence, for $j\in B$,

$$\begin{aligned}
\sum_{\alpha,\beta=1}^n F^{\alpha\beta}a_{jj,\alpha\beta}= & \sum_{\alpha,\beta=1}^n\frac{F^{\alpha\beta}}{u_n^3}[-u_n^2u_{\alpha\beta jj}-4u_{n\alpha}u_{nj}u_{j\beta}+4u_nu_{j\alpha}u_{nj\beta}\\
& +2u_nu_{nj}u_{\alpha\beta j}-2u_{nn}u_{j\alpha}u_{j\beta}]+O(\mathcal{H}_\phi).
\end{aligned}\tag{2.19}$$

Using the fact that $\sum_{\alpha=1}^n F^{\alpha n}u_{n\alpha}=(\sum_{\alpha,\beta=1}^n-\sum_{\beta=1}^{n-1}\sum_{\alpha=1}^n)F^{\alpha\beta}u_{\alpha\beta}$, $\forall j\in B$,

$$\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{n\alpha}u_{j\beta}=u_{nj}(\sum_{\alpha,\beta=1}^n-\sum_{\beta=1}^{n-1}\sum_{\alpha=1}^n)F^{\alpha\beta}u_{\alpha\beta}+O(\mathcal{H}_\phi),$$

$$\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{j\alpha}u_{nj\beta}=u_{nj}(\sum_{\alpha,\beta=1}^n-\sum_{\alpha=1}^{n-1}\sum_{\beta=1}^n)F^{\alpha\beta}u_{\alpha\beta j}+O(\mathcal{H}_\phi),$$

and

$$\begin{aligned}
& -2u_{nn}\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{j\alpha}u_{j\beta}=-2u_{nn}F^{nn}u_{nj}^2+O(\mathcal{H}_\phi)\\
= & -2u_{nj}^2\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{\alpha\beta}+4u_{nj}^2\sum_{\alpha=1}^{n-1}F^{\alpha n}u_{n\alpha}+2u_{nj}^2\sum_{\alpha,\beta=1}^{n-1}F^{\alpha\beta}u_{\alpha\beta}+O(\mathcal{H}_\phi).
\end{aligned}$$

Put above to (2.19),

$$
\begin{aligned}
&\sum_{j\in B}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_n^3 a_{jj,\alpha\beta}\\
= &-u_n^2\sum_{j\in B}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta jj} + 6u_n\sum_{j\in B}u_{nj}\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta j}\\
&-6\sum_{j\in B}u_{nj}^2\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta} - 4u_n\sum_{j\in B}u_{nj}\sum_{\alpha=1}^{n-1}\sum_{\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta j}\\
&+8\sum_{j\in B}u_{nj}^2\sum_{\alpha=1}^{n-1}F^{\alpha n}u_{n\alpha} + 6\sum_{j\in B}u_{nj}^2\sum_{\alpha,\beta=1}^{n-1}F^{\alpha\beta}u_{\alpha\beta} + O(\mathcal{H}_\phi).
\end{aligned}
\tag{2.20}
$$

By (2.17), for $j \in B$,

$$
\begin{aligned}
&u_n\sum_{\alpha=1}^{n-1}\sum_{\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta j} = u_n\sum_{\alpha=1}^{n}\left(\sum_{i\in B}F^{\alpha i}u_{ij\alpha} + \sum_{i\in G}F^{\alpha i}u_{ij\alpha}\right)\\
= &\sum_{\alpha=1}^{n}\sum_{i\in G}F^{\alpha i}(-u_n^2 a_{ij,\alpha} + u_{i\alpha}u_{jn} + u_{j\alpha}u_{in})\\
&+\sum_{\alpha=1}^{n}\sum_{i\in B}F^{\alpha i}(u_{i\alpha}u_{jn} + u_{j\alpha}u_{in}) + O(\mathcal{H}_\phi)\\
= &-u_n^2\sum_{\alpha=1}^{n}\sum_{i\in G}F^{\alpha i}a_{ij,\alpha} + u_{nj}\sum_{i\in G}F^{ii}u_{ii} + 2u_{nj}(\sum_{i=1}^{n-1}F^{ni}u_{ni}) + O(\mathcal{H}_\phi).
\end{aligned}
\tag{2.21}
$$

(2.20) and (2.21) yield

$$
\begin{aligned}
\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_n^3 a_{jj,\alpha\beta} = &-u_n^2\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta jj} + 2u_n u_{nj}\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta j}\\
&+4u_n u_{nj}\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta j} - 6u_{nj}^2\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta}\\
&+4u_n^2 u_{nj}\sum_{\alpha=1}^{n}\sum_{i\in G}F^{\alpha i}a_{ij,\alpha} + 2u_{nj}^2\sum_{i\in G}F^{ii}u_{ii} + O(\mathcal{H}_\phi).
\end{aligned}
\tag{2.22}
$$

So,

$$
\begin{aligned}
&\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\phi_{\alpha\beta} \\
= \;& u_n^{-3}\sum_{j\in B}\left[\sigma_l(G)+\frac{\sigma_1^2(B|j)-\sigma_2(B|j)}{\sigma_1^2(B)}\right][-u_n^2\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta jj}+2u_nu_{nj}\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta j} \\
&\qquad\qquad +4u_nu_{nj}\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta j}-6u_{nj}^2\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}u_{\alpha\beta}] \\
&-2\sum_{j\in B,i\in G}\left[\sigma_l(G)+\frac{\sigma_1^2(B|j)-\sigma_2(B|j)}{\sigma_1^2(B)}\right]\left[\sum_{\alpha,\beta=1}^{n}F^{\alpha\beta}\frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}}-2\frac{u_{nj}}{u_n}\sum_{\alpha=1}^{n}F^{\alpha i}a_{ij,\alpha}-\frac{u_{nj}^2}{u_n^3}F^{ii}u_{ii}\right] \\
&-\frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\in B}F^{\alpha\beta}[\sigma_1(B)a_{ii,\alpha}-a_{ii}\sum_{j\in B}a_{jj,\alpha}][\sigma_1(B)a_{ii,\beta}-a_{ii}\sum_{j\in B}a_{jj,\beta}] \\
&-\frac{1}{\sigma_1(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\neq j\in B}F^{\alpha\beta}a_{ij,\alpha}a_{ij,\beta}+O(\mathcal{H}_\phi).
\end{aligned}
$$

In fact, for any $i\in G, j\in B$,

$$
\begin{aligned}
&\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}} - 2\frac{u_{nj}}{u_n}\sum_{\alpha=1}^{n} F^{\alpha i}a_{ij,\alpha} - \frac{u_{nj}^2}{u_n^3}F^{ii}u_{ii}\\
= &\ -\frac{1}{u_n^3}[\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\frac{h_{ij,\alpha}h_{ij,\beta}}{h_{ii}} - 2\frac{u_{nj}}{u_n}\sum_{\alpha=1}^{n} F^{\alpha i}h_{ij,\alpha} + u_{nj}^2F^{ii}u_{ii}]\\
= &\ -\frac{1}{u_n^3}\Bigg\{\sum_{\alpha,\beta=1}^{n-1} F^{\alpha\beta}\frac{1}{u_n^2u_{ii}}[u_n^2u_{ij\alpha} - u_nu_{i\alpha}u_{jn}][u_n^2u_{ij\beta} - u_nu_{i\beta}u_{jn}]\\
&\quad +2\sum_{\alpha=1}^{n-1} F^{\alpha n}\frac{1}{u_n^2u_{ii}}[u_n^2u_{ij,\alpha} - u_nu_{i\alpha}u_{jn}][u_n^2u_{ijn} - 2u_nu_{in}u_{jn}]\\
&\quad +F^{nn}\frac{1}{u_n^2u_{ii}}[u_n^2u_{ijn} - 2u_nu_{in}u_{jn}][u_n^2u_{ijn} - 2u_nu_{in}u_{jn}]\\
&\quad -2\sum_{\alpha=1}^{n-1} F^{\alpha i}\frac{1}{u_n^2u_{ii}}[u_n^2u_{ij\alpha} - 2u_nu_{i\alpha}u_{jn}][u_nu_{ii}u_{nj}]\\
&\quad -2F^{ni}\frac{1}{u_n^2u_{ii}}[u_n^2u_{ijn} - 2u_nu_{in}u_{jn}][u_nu_{ii}u_{nj}]\\
&\quad +F^{ii}\frac{1}{u_n^2u_{ii}}(u_nu_{ii}u_{nj})^2\Bigg\}\\
= &\ -\frac{1}{u_n^3}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\frac{1}{u_n^2u_{ii}}[u_n^2u_{ij\alpha} - 2u_nu_{i\alpha}u_{jn}][u_n^2u_{ij\beta} - 2u_nu_{i\beta}u_{jn}]. \qquad (2.23)
\end{aligned}
$$

Obviously, we can get

$$
\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}} - 2\frac{u_{nj}}{u_n}\sum_{\alpha=1}^{n} F^{\alpha i}a_{ij,\alpha} - \frac{u_{nj}^2}{u_n^3}F^{ii}u_{ii} \geqslant 0, \qquad (2.24)
$$

this is the Claim in [3].

From the above formulas, Lemma 2.1 holds. □

2.2. **Proof of Theorem 1.1.** We start this section with a discussion on structure condition (1.3). For any function $F(r,p,u,t)$, denote $F^{\alpha\beta} = \frac{\partial F}{\partial r_{\alpha\beta}}, F^{p_l} = \frac{\partial F}{\partial u_l}, \cdots$ as partial derivatives of F with respect to corresponding arguments.

Lemma 2.2. *If F satisfies condition* (1.3)*, then*

$$
\begin{aligned}
Q(V,V) &= F^{\alpha\beta,\gamma\eta}X_{\alpha\beta}X_{\gamma\eta} + 2F^{\alpha\beta,p_l}\theta_lX_{\alpha\beta}Y + F^{p_k,p_l}\theta_k\theta_lY^2\\
&\quad +4s^{-1}F^{\alpha\beta}X_{\alpha\beta}Y - 6F^{\alpha\beta}A_{\alpha\beta}Y^2\\
&\leqslant 0, \qquad (2.25)
\end{aligned}
$$

for every $(X_{\alpha\beta}, Y) = ((s^2\widetilde{X}_{\alpha\beta} + 2sA_{\alpha\beta}\widetilde{Y}), \widetilde{Y})$*, with any* $\widetilde{V} = ((\widetilde{X}_{\alpha\beta}), \widetilde{Y}) \in \mathcal{S}^n \times \mathbb{R}$*, where* $F^{\alpha\beta,rs}, F^{\alpha\beta,u_l}$*, etc. are evaluated at* $(s^2A, s\theta, u, t)$*, and the Einstein summation convention is used.*

Proof: Denoting $\widetilde{F}(A,s) = F(s^2A, s\theta, u, t)$, condition (1.3) implies that $\widetilde{F}(A,s)$ is locally concave, that is,

$$\widetilde{F}^{\alpha\beta,\gamma\eta}\widetilde{X}_{\alpha\beta}\widetilde{X}_{\gamma\eta} + 2\widetilde{F}^{\alpha\beta,s}\widetilde{X}_{\alpha\beta}\widetilde{Y} + \widetilde{F}^{s,s}\widetilde{Y}^2 \leqslant 0, \tag{2.26}$$

for any $\widetilde{V} = ((\widetilde{X}_{\alpha\beta}), \widetilde{Y}) \in \mathcal{S}^n \times \mathbb{R}$.

At (A, s),

$$\begin{aligned}
\widetilde{F}^{\alpha\beta,\gamma\eta} &= F^{\alpha\beta,\gamma\eta}s^2 \cdot s^2,\\
\widetilde{F}^{\alpha\beta,s} &= F^{\alpha\beta,\gamma\eta}s^2 \cdot 2sA_{\gamma\eta} + F^{\alpha\beta,p_l}s^2 \cdot \theta_l + F^{\alpha\beta}2s,\\
\widetilde{F}^{s,s} &= F^{\alpha\beta,\gamma\eta}2sA_{\alpha\beta} \cdot 2sA_{\gamma\eta} + 2F^{\alpha\beta,p_l}2sA_{\alpha\beta} \cdot \theta_l + F^{p_k,p_l}\theta_k \cdot \theta_l + F^{\alpha\beta}2A_{\alpha\beta}.
\end{aligned}$$

Set

$$X_{\alpha\beta} = s^2\widetilde{X}_{\alpha\beta} + 2sA_{\alpha\beta}\widetilde{Y}, \tag{2.27}$$

$$Y = \widetilde{Y}, \tag{2.28}$$

so (2.26) is equivalent to

$$\begin{aligned}
& F^{\alpha\beta,\gamma\eta}X_{\alpha\beta}X_{\gamma\eta} + 2F^{\alpha\beta,p_l}\theta_l X_{\alpha\beta}Y + F^{u_k,p_l}\theta_k\theta_l Y^2\\
& +4s^{-1}F^{\alpha\beta}s^2\widetilde{X}_{\alpha\beta}\widetilde{Y} + 2F^{\alpha\beta}A_{\alpha\beta}\widetilde{Y}^2\\
= \ & F^{\alpha\beta,\gamma\eta}X_{\alpha\beta}X_{\gamma\eta} + 2F^{\alpha\beta,p_l}\theta_l X_{\alpha\beta}Y + F^{p_k,p_l}\theta_k\theta_l Y^2\\
& +4s^{-1}F^{\alpha\beta}X_{\alpha\beta}Y - 6F^{\alpha\beta}A_{\alpha\beta}Y^2\\
\leqslant \ & 0.
\end{aligned}$$

Therefore, (2.25) follows from above, and Lemma 2.2 holds. □

Theorem 1.1 is a direct consequence of the following proposition and the strong maximum principle.

Proposition 2.3. Suppose that the function F, u satisfy assumptions in Theorem 1.1. If the second fundamental form b_{ij} of Σ^{c,t_0} attains minimum rank $l = l(t_0)$ at certain point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O} \times (t_0 - \delta_0, t_0 + \delta_0]$ of (x_0, t_0) and a positive constant C independent of ϕ (defined in (2.9)), such that

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\phi_{\alpha\beta}(x,t) - \phi_t \leqslant C(\phi + |\nabla\phi|), \ \forall\ (x,t) \in \mathcal{O} \times (t_0 - \delta_0, t_0 + \delta_0]. \tag{2.29}$$

Proof: Let $u \in C^{3,1}(\Omega \times [0,T])$ be a spatial quasiconcave solution of equation (1.1) and $(u_{ij}) \in \mathcal{S}^n$. Let $l = l(t_0)$ be the minimum rank of the second fundamental forms h_{ij} of Σ^{c,t_0} ($l \in \{0, 1, ..., n-1\}$) for every c in $(-\gamma_0 + c_0, \gamma_0 + c_0)$, suppose the minimum rank l arrives at point $x_0 \in \Sigma^{c,t_0}$. We work on a small open neighborhood $\mathcal{O} \times (t_0 - \delta_0, t_0 + \delta_0]$ of (x_0, t_0). We may assume $l \leqslant n-2$. Lemma 2.1 implies $\phi \in C^{1,1}(\mathcal{O} \times (t_0 - \delta_0, t_0 + \delta_0])$, $\phi(x,t) \geqslant 0, \quad \phi(x_0, t_0) = 0$. For $\epsilon > 0$ sufficient small, let ϕ_ϵ defined as in (2.9) and

(2.10), we need to verify (2.29) for each point $(x,t)\in\mathcal{O}\times(t_0-\delta_0,t_0+\delta_0]$. For each fixed (x,t), choose a local coordinate $e_1,\cdots,e_{n-1},e_n$ such that (2.6) and (2.7) are satisfied. We want to establish differential inequality (2.29) for ϕ_ε defined in (2.10) with constant C independent of ε. Note that we will omit the subindex ε with the understanding that all the estimates are independent of ε.

By Lemma 2.1,

$$
\begin{aligned}
&\sum_{\alpha,\beta=1}^n F^{\alpha\beta}\phi_{\alpha\beta}-\phi_t\\
\leqslant\;& -u_n^{-3}\sum_{j\in B}\left[\sigma_l(G)+\frac{\sigma_1^2(B|j)-\sigma_2(B|j)}{\sigma_1^2(B)}\right]\Big[u_n^2(\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{jj\alpha\beta}-u_{jjt})\\
&-2u_nu_{jn}(\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{j\alpha\beta}-u_{jt})-4u_nu_{jn}\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{j\alpha\beta}+6u_{jn}^2\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{\alpha\beta}\Big]\\
&-\frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta=1}^n\sum_{i\in B}F^{\alpha\beta}[\sigma_1(B)a_{ii,\alpha}-a_{ii}\sum_{j\in B}a_{jj,\alpha}][\sigma_1(B)a_{ii,\beta}-a_{ii}\sum_{j\in B}a_{jj,\beta}]\\
&-\frac{1}{\sigma_1(B)}\sum_{\alpha,\beta=1}^n\sum_{i\neq j,i,j\in B}F^{\alpha\beta}a_{ij,\alpha}a_{ij,\beta}+O(\mathcal{H}_\phi).
\end{aligned}\tag{2.30}
$$

For each $j\in B$, differentiating equation (1.1) in e_j direction at x,

$$
u_{jt}=\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{\alpha\beta j}+F^{u_n}u_{jn}+O(\mathcal{H}_\phi),\tag{2.31}
$$

and

$$
\begin{aligned}
u_{jjt}\;=\;&\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{\alpha\beta jj}+\sum_{\alpha,\beta,r,s=1}^n F^{\alpha\beta,rs}u_{\alpha\beta j}u_{rsj}+2\sum_{\alpha,\beta,l=1}^n F^{\alpha\beta,u_l}u_{\alpha\beta j}u_{lj}\\
&+2\sum_{\alpha,\beta=1}^n F^{\alpha\beta,u}u_{j\alpha\beta}u_j+\sum_{l,s=1}^n F^{u_l,u_s}u_{lj}u_{sj}-2\sum_{l=1}^n F^{u_l,u}u_{lj}u_j\\
&+F^{u,u}u_j^2+\sum_{l=1}^n F^{u_l}u_{ljj}+F^uu_{jj}.
\end{aligned}\tag{2.32}
$$

It follows from (2.17) that, at (x,t)

$$
\begin{aligned}
\sum_{\alpha,\beta=1}^n F^{\alpha\beta}u_{\alpha\beta jj}-u_{jjt}\;=\;&-\sum_{\alpha,\beta,r,s=1}^n F^{\alpha\beta,rs}u_{\alpha\beta j}u_{rsj}-2\sum_{\alpha,\beta=1}^n F^{\alpha\beta,u_n}u_{j\alpha\beta}u_{nj}\\
&-F^{u_n,u_n}u_{jn}^2-2\frac{F^{u_n}}{u_n}u_{jn}^2+O(\mathcal{H}_\phi).
\end{aligned}\tag{2.33}
$$

Since $u_{\alpha\beta jj} = u_{jj\alpha\beta}$, (2.31) and (2.33) yield

$$
\begin{aligned}
& F^{\alpha\beta}\phi_{\alpha\beta} - \phi_t \\
\leqslant\ & \sum_{j\in B} u_n^{-3}\left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right]\Bigg\{\Big[\sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}u_{\alpha\beta j}u_{rsj} \\
& \qquad + 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,u_n}u_{j\alpha\beta}u_{jn} + F^{u_n,u_n}u_{jn}^2\Big]u_n^2 \\
& \qquad + 4u_{jn}u_n\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{j\alpha\beta} - 6u_{jn}^2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta}\Bigg\} \\
& - \frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\in B} F^{\alpha\beta}[\sigma_1(B)a_{ii,\alpha} - a_{ii}\sum_{j\in B} a_{jj,\alpha}][\sigma_1(B)a_{ii,\beta} - a_{ii}\sum_{j\in B} a_{jj,\beta}] \\
& - \frac{1}{\sigma_1(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\neq j, i,j\in B} F^{\alpha\beta}a_{ij,\alpha}a_{ij,\beta} + O(\mathcal{H}_\phi).
\end{aligned} \tag{2.34}
$$

For each $j \in B$, set

$$
\begin{aligned}
S_j = \ & \Big[\sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}u_{j\alpha\beta}u_{rsj} + 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,u_n}u_{j\alpha\beta}u_{jn} + F^{u_n,u_n}u_{jn}^2\Big]u_n^2 \\
& + \ 4\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{j\alpha\beta}u_{jn}u_n - 6\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta}u_{jn}^2
\end{aligned} \tag{2.35}
$$

For each $j \in B$, set

$$X_{\alpha\beta} = u_{\alpha\beta j}u_n, \forall(\alpha,\beta), \tag{2.36}$$

$$Y = u_{jn}u_n. \tag{2.37}$$

In the coordinate system (2.6),

$$(\nabla^2 u(x), \nabla u(x), u(x), t) = (\nabla^2 u, (0, ..., 0, |\nabla u|), u, t).$$

Equalize it to $(s^2 A, s\theta, u, t)$, the components of $\widetilde{V}$ defined in Lemma 2.2 are

$$
\begin{aligned}
\widetilde{X}_{\alpha\beta} &= \frac{u_{\alpha\beta j}}{u_n} - \frac{2u_{\alpha\beta}u_{jn}}{u_n^2}, \quad \forall(\alpha,\beta), \\
\widetilde{Y} &= u_{jn}u_n.
\end{aligned}
$$

For $j \in B$, Lemma 2.2 implies

$$S_j \leqslant 0. \tag{2.38}$$

Condition (1.2) implies

$$(F^{\alpha\beta}) \geqslant \delta_0 I, \quad \text{for some } \delta_0 > 0, \text{ and } \forall x \in \mathcal{O}. \tag{2.39}$$

Set

$$V_{i\alpha} = \sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\alpha}.$$

Combining (2.34), (2.38) and (2.39),

$$F^{\alpha\beta}\phi_{\alpha\beta} \leqslant C(\phi + \sum_{i,j\in B} |\nabla a_{ij}|) - \delta_0 [\frac{\sum_{i\neq j\in B,\alpha=1}^n a_{ij\alpha}^2}{\sigma_1(B)} + \frac{\sum_{i\in B,\alpha=1}^n V_{i\alpha}^2}{\sigma_1^3(B)}]. \tag{2.40}$$

By Lemma 3.3 in [2], for each $M \geqslant 1$, for any $M \geqslant |\gamma_i| \geqslant \frac{1}{M}$, there is a constant C depending only on n and M such that, $\forall \alpha$,

$$\sum_{i,j\in B} |a_{ij\alpha}| \leqslant C(1 + \frac{1}{\delta_0^2})(\sigma_1(B) + |\sum_{i\in B} \gamma_i a_{ii\alpha}|) + \frac{\delta_0}{2}[\frac{\sum_{i\neq j\in B} |a_{ij\alpha}|^2}{\sigma_1(B)} + \frac{\sum_{i\in B} V_{i\alpha}^2}{\sigma_1^3(B)}]. \tag{2.41}$$

Taking $\gamma_i = \sigma_l(G) + \frac{\sigma_1^2(B|i) - \sigma_2(B|i)}{\sigma_1^2(B)}$ for each $i \in B$, the Newton-MacLaurine inequality implies

$$\sigma_l(G) + 1 \geqslant \sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \geqslant \sigma_l(G), \quad \forall j \in B.$$

Therefore we conclude from Lemma 2.1 and (2.41) that $\sum_{i,j\in B} |\nabla a_{ij}|$ can be controlled by the rest terms on the right hand side in (2.40) and $\phi + |\nabla \phi|$. The proof is complete. □

3. Proof of Theorem 1.2

In this section, through modifying the proof of Theorem 1.1, we will give a proof of Theorem 1.2. Also it is a parabolic equation case corresponding to [11].

Suppose that $u(x,t)$ is a spatial quasiconcave solution of (1.4), and assume that level surface $\Sigma^{u(x_0,t)} = \{x \in \Omega | u(x,t) = u(x_0,t)\}$ is connected for each $(x_0,t) \in \mathcal{O} \times [0,T]$.

Set

$$\widetilde{a} = a - \eta_0 g I, \quad \eta_0 \geqslant 0, \quad g(x,t) = e^{Au(x,t)}, \tag{3.1}$$

where $A > 0$ is a constant to be determined. We want to show $\widetilde{a}$ is of constant rank. Theorem 1.1 corresponds to the case $\eta_0 = 0$. If $\min\{\kappa^0, \kappa^1\} = 0$, there is nothing to prove instead of utilizing Theorem 1.1. We will assume $\min\{\kappa^0, \kappa^1\} > 0$ in the rest of the paper. Denote $\kappa_s(x,t)$ and $\widetilde{\kappa}_s(x,t)$ be the minimum eigenvalue of matrix $a(x)$ and $\widetilde{a}(x)$ respectively. Since the spatial level sets are strictly convex, and $\overline{\Omega}$ is compact, a is strictly positive definite. That is, $\kappa_s(x,t)$ has a positive lower bound.

For a positive constant A to be determined, increasing η_0 from 0, such that $\widetilde{a}$ is degenerate at some points, i.e. $\widetilde{a}$ is semi-positive with the rank is not full. (1.8) follows easily if this happens only on the boundary. We want to show that, if the degeneracy happens at an interior point of Ω, then $\widetilde{a}$ is degenerate through out Ω with the same rank. This implies that the "=" holds in (1.8) and Theorem 1.2 is proved.

Therefore, the main task is to prove constant rank theorem for $\widetilde{a}$. Suppose $\widetilde{a}(x,t_0)$ attains minimal rank $l = l(t_0)$ at some point $z_0 \in \Omega$. We may assume $l \leqslant n-2$, otherwise there is nothing to prove. And we assume $u \in C^{3,1}$ and $u_n > 0$ in the rest of this paper. So there is a neighborhood $\mathcal{O} \times (t_0 - \delta, t_0 + \delta]$ of (z_0, t_0), such that there are l

"good" eigenvalues of $(\widetilde{a}_{ij})$ which are bounded below by a positive constant, and the other $n-1-l$ "bad" eigenvalues of $(\widetilde{a}_{ij})$ are very small. Denote G be the index set of these "good" eigenvalues and B be the index set of "bad" eigenvalues. And for any fixed point $(x,t)\in\mathcal{O}\times(t_0-\delta,t_0+\delta]$, we may express $(\widetilde{a}_{ij})$ in a form of (3.1) and (2.5), by choosing $e_1,\cdots,e_{n-1},e_n$ such that

$$|\nabla u(x,t)|=u_n(x,t)>0 \text{ and } (u_{ij}), i,j=1,..,n-1, \text{ is diagonal at } (x,t). \tag{3.2}$$

Without loss of generality, we assume $u_{11}\geqslant u_{22}\geqslant\cdots\geqslant u_{n-1,n-1}$. So, at $(x,t)\in\mathcal{O}\times(t_0-\delta,t_0+\delta)$, from (2.5), we have the matrix $(a_{ij}), i,j=1,..,n-1$, is also diagonal. And without loss of generality we may assume $a_{11}\geqslant a_{22}\geqslant ...\geqslant a_{n-1,n-1}$, then $\widetilde{a}_{11}\geqslant\widetilde{a}_{22}\geqslant ...\geqslant\widetilde{a}_{n-1,n-1}$. There is a positive constant $C>0$ depending only on $\|u\|_{C^4}$ and $\mathcal{O}\times(t_0-\delta,t_0+\delta]$, such that $\widetilde{a}_{11}\geqslant\widetilde{a}_{22}\geqslant...\geqslant\widetilde{a}_{ll}>C$ for all $(x,t)\in\mathcal{O}\times(t_0-\delta,t_0+\delta)$. For convenience we denote $G=\{1,\cdots,l\}$ and $B=\{l+1,\cdots,n-1\}$ be the "good" and "bad" sets of indices respectively. If there is no confusion, we also denote

$$G=\{\widetilde{a}_{11},...,\widetilde{a}_{ll}\} \textit{ and } B=\{\widetilde{a}_{l+1,l+1},...,\widetilde{a}_{n-1,n-1}\}. \tag{3.3}$$

Note that for any $\delta>0$, we may choose $\mathcal{O}\times(t_0-\delta,t_0+\delta]$ small enough such that $\widetilde{a}_{jj}<\delta$ for all $j\in B$ and $(x,t)\in\mathcal{O}\times(t_0-\delta,t_0+\delta]$.

For each (x,t), let $a=(a_{ij})$ be the symmetric Weingarten tensor of $\Sigma^{u(x,t)}$. Set

$$p(\widetilde{a})=\sigma_{l+1}(\widetilde{a}_{ij}),\quad q(\widetilde{a})\ =\ \begin{cases}\frac{\sigma_{l+2}(\widetilde{a}_{ij})}{\sigma_{l+1}(\widetilde{a}_{ij})}, & \text{if } \sigma_{l+1}(\widetilde{a}_{ij})>0,\\ 0, & \text{otherwise.}\end{cases} \tag{3.4}$$

Theorem 1.2 is equivalent to say $p(\widetilde{a})\equiv 0$ (defined in (3.4)) in $\mathcal{O}\times(t_0-\delta,t_0]$. As in the description of the proof of Theorem 1.1, we should consider the function

$$\phi(\widetilde{a})=p(\widetilde{a})+q(\widetilde{a}), \tag{3.5}$$

where p and q as in (3.4). We will use notion $h=O(f)$ if $|h(x,t)|\leqslant Cf(x,t)$ for $(x,t)\in\mathcal{O}\times(t_0-\delta,t_0+\delta]$ with positive constant C under control.

To get around $p=0$, for $\varepsilon>0$ sufficiently small, we instead consider

$$\phi_\varepsilon(\widetilde{a})=\phi(\widetilde{a}_\varepsilon), \tag{3.6}$$

where $a_\varepsilon=\widetilde{a}+\varepsilon I$. We will also denote $G_\varepsilon=\{\widetilde{a}_{ii}+\varepsilon, i\in G\}$, $B_\varepsilon=\{\widetilde{a}_{ii}+\varepsilon, i\in B\}$.

To simplify the notations, we will drop subindex ε with the understanding that all the estimates will be independent of ε. In this setting, if we pick $\mathcal{O}\times(t_0-\delta,t_0+\delta]$ small enough, there is $C>0$ independent of ε such that

$$\phi(\widetilde{a}(x,t))\geqslant C\varepsilon,\quad \sigma_1(B)\geqslant C\varepsilon,\quad \text{for all (x,t)}\in\mathcal{O}\times(\mathrm{t_0}-\delta,\mathrm{t_0}+\delta]. \tag{3.7}$$

We also denote

$$\mathcal{H}_\phi=\sum_{i,j\in B}|\nabla\widetilde{a}_{ij}|+\phi. \tag{3.8}$$

Lemma 3.1. *For any fixed $(x,t)\in\mathcal{O}\times(t_0-\delta,t_0+\delta]$, with the coordinate chosen as in (3.2) and (3.3),*

$$
\begin{aligned}
&\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\phi_{\alpha\beta} - \phi_t \\
= \quad & u_n^{-3}\sum_{j\in B}\left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right][-u_n^2(\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta jj} - u_{jjt}) + 2u_nu_{nj}(\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta j} - u_{jt}) \\
&\qquad + 4u_nu_{nj}\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta j} - 6u_{nj}^2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta}] \\
&+2u_n^{-3}\sum_{j\in B, i\in G}\left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right]\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\frac{1}{u_{ii}}[u_nu_{ij\alpha} - 2u_{i\alpha}u_{jn}][u_nu_{ij\beta} - 2u_{i\beta}u_{jn}] \\
&+\eta_0 g\left[-A^2F^{nn}u_n^2 + AO(1) + O(1)\right] \\
&-\frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\in B} F^{\alpha\beta}[\sigma_1(B)\widetilde{a}_{ii,\alpha} - \widetilde{a}_{ii}\sum_{j\in B}\widetilde{a}_{jj,\alpha}][\sigma_1(B)\widetilde{a}_{ii,\beta} - \widetilde{a}_{ii}\sum_{j\in B}\widetilde{a}_{jj,\beta}] \\
&-\frac{1}{\sigma_1(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\neq j\in B} F^{\alpha\beta}\widetilde{a}_{ij,\alpha}\widetilde{a}_{ij,\beta} + O(\mathcal{H}_\phi).
\end{aligned}
$$

Proof: For any fixed $(x,t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta]$, we choose the coordinate as in (3.2) such that $|\nabla u(x)| = u_n(x) > 0$ and the matrix $(\widetilde{a}_{ij}(x))$ is diagonal for $1 \leqslant i,j \leqslant n-1$ and nonnegative. From the definition of p,

$$
a_{jj} = -\frac{h_{jj}}{u_n^3} = -\frac{u_{jj}}{u_n} = O(\mathcal{H}_\phi), \forall j \in B, \tag{3.9}
$$

and

$$
\phi_t = \sum_{j\in B}\left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right]\widetilde{a}_{jjt} + O(\mathcal{H}_\phi). \tag{3.10}
$$

Using relationship (3.9), we have

$$
\begin{aligned}
\phi_{\alpha\beta} \quad = \quad & \sum_{j\in B}\left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right]\left[\widetilde{a}_{jj,\alpha\beta} - 2\sum_{i\in G}\frac{\widetilde{a}_{ij,\alpha}\widetilde{a}_{ij,\beta}}{\widetilde{a}_{ii}}\right] \\
& -\frac{1}{\sigma_1^3(B)}\sum_{i\in B}\left[\sigma_1(B)\widetilde{a}_{ii,\alpha} - \widetilde{a}_{ii}\sum_{j\in B}\widetilde{a}_{jj,\alpha}\right]\left[\sigma_1(B)\widetilde{a}_{ii,\beta} - \widetilde{a}_{ii}\sum_{j\in B}\widetilde{a}_{jj,\beta}\right] \\
& -\frac{1}{\sigma_1(B)}\sum_{i\neq j\in B}\widetilde{a}_{ij,\alpha}\widetilde{a}_{ij,\beta} + O(\mathcal{H}_\phi).
\end{aligned} \tag{3.11}
$$

So,

$$
\begin{aligned}
&\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}[\widetilde{a}_{jj,\alpha\beta} - 2\sum_{i\in G}\frac{\widetilde{a}_{ij,\alpha}\widetilde{a}_{ij,\beta}}{\widetilde{a}_{ii}}] - \widetilde{a}_{jj,t} \\
= &\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} a_{jj,\alpha\beta} - a_{jj,t} + \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}(-\eta_0 g_{\alpha\beta}) + \eta_0 g_t \\
&-2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\sum_{i\in G}\frac{\widetilde{a}_{ij,\alpha}\widetilde{a}_{ij,\beta}}{\widetilde{a}_{ii}}.
\end{aligned}
\tag{3.12}
$$

From the definition of a_{ij}, and $u_k = 0$ for $k = 1, \cdots, n-1$, we can get

$$
u_n u_{ij\alpha} = -u_n^2 a_{ij,\alpha} + u_{nj}u_{i\alpha} + u_{ni}u_{j\alpha} + u_{n\alpha}u_{ij}
\tag{3.13}
$$

and

$$
\begin{aligned}
u_n^3 a_{jj,\alpha\beta} = &-u_n^2 u_{jj\alpha\beta} + 2u_n u_{nj} u_{\alpha\beta j} - 2u_n(u_{n\beta}u_{jj\alpha} + u_{n\alpha}u_{jj\beta}) \\
&+2u_n(u_{j\alpha}u_{nj\beta} + u_{j\beta}u_{nj\alpha}) + 2u_{nj}(u_{n\alpha}u_{j\beta} + u_{n\beta}u_{j\alpha}) - 2u_{nn}u_{j\alpha}u_{j\beta} \\
&-2(u_{n\alpha}u_{n\beta} + u_n u_{\alpha\beta n})u_{jj} - 2\eta_0 g u_{j\alpha}u_{j\beta}u_n - 3\eta_0 u_n^2(u_{n\alpha}g_\beta + u_{n\beta}g_\alpha) \\
&-\eta_0 g(3u_n^2 u_{n\alpha\beta} + 6u_{n\alpha}u_{n\beta}u_n + \sum_{i=1}^{n-1} u_{i\alpha}u_{i\beta}u_n) + O(\mathcal{H}_\phi).
\end{aligned}
\tag{3.14}
$$

Direct calculation and (3.13), we can get

$$
\begin{aligned}
&-a_{jj,t} + \sum_{\alpha\beta=1}^{n} F^{\alpha\beta}(-\eta_0 g_{\alpha\beta}) + \eta_0 g_t \\
= &\frac{1}{u_n^3}[u_n^2 u_{jjt} - 2u_n u_{nj} u_{jt}] \\
&+\eta_0 g[-A^2 F^{nn} u_n^2 - A(\sum_{\alpha\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta} - u_t) + \frac{u_{nt}}{u_n}].
\end{aligned}
\tag{3.15}
$$

From (3.14),

$$
\begin{aligned}
\sum_{\alpha\beta=1}^{n} F^{\alpha\beta} a_{jj,\alpha\beta} = &\sum_{\alpha\beta=1}^{n} \frac{F^{\alpha\beta}}{u_n^3}[-u_n^2 u_{jj\alpha\beta} + 2u_n u_{nj} u_{\alpha\beta j} \\
&-4u_{nj}u_{n\alpha}u_{j\beta} + 4u_{nj}u_{n\alpha}u_{j\beta} - 2u_{nn}u_{j\alpha}u_{j\beta} \\
&-2\eta_0 u_n^2 u_{n\alpha} g_\beta - \eta_0 g(u_n^2 u_{n\alpha\beta} + 2u_{j\alpha}u_{j\beta}u_n + \sum_{i=1}^{n-1} u_{i\alpha}u_{i\beta}u_n)] + O(\mathcal{H}_\phi),
\end{aligned}
$$

so, as in [11], we can get

$$
\begin{aligned}
& u_n^3 \sum_{\alpha\beta=1}^{n} F^{\alpha\beta} a_{jj,\alpha\beta} \\
= \; & -u_n^2 \sum_{\alpha\beta=1}^{n} F^{\alpha\beta} u_{jj\alpha\beta} + 2u_n u_{nj} \sum_{\alpha\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta j} \\
& +4u_n u_{nj} \sum_{\alpha\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta j} - 6u_{nj}^2 \sum_{\alpha\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta} \\
& +4u_n^2 \sum_{\alpha=1}^{n} \sum_{i\in G} F^{\alpha i} a_{ij,\alpha} + 2u_{nj}^2 \sum_{i\in G} F^{ii} u_{ii} \\
& +2u_{nj}^2 \sum_{i\in B} F^{ii} u_{ii} - 12u_{jn} u_{jj} \sum_{\alpha=1}^{n} F^{j\alpha} u_{n\alpha} + 4u_n u_{jj} \sum_{\alpha=1}^{n} F^{j\alpha} u_{jn\alpha} - 2u_{nn} F^{jj} u_{jj}^2 \\
& -\eta_0 g (u_n^2 u_{n\alpha\beta} + 2u_{j\alpha} u_{j\beta} u_n + \sum_{i=1}^{n-1} u_{i\alpha} u_{i\beta} u_n) \\
& -2\eta_0 \sum_{\alpha\beta=1}^{n} F^{\alpha\beta} u_{n\alpha} g_\beta u_n + 4\eta_0 \sum_{\alpha=1}^{n} F^{j\alpha} g_\alpha u_{jn} u_n^2 + O(\mathcal{H}_\phi), \\
= \; & -u_n^2 \sum_{\alpha\beta=1}^{n} F^{\alpha\beta} u_{jj\alpha\beta} + 2u_n u_{nj} \sum_{\alpha\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta j} \\
& +4u_n u_{nj} \sum_{\alpha\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta j} - 6u_{nj}^2 \sum_{\alpha\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta} \\
& +4u_n^2 \sum_{\alpha=1}^{n} \sum_{i\in G} F^{\alpha i} a_{ij,\alpha} + 2u_{nj}^2 \sum_{i\in G} F^{ii} u_{ii} \\
& +\eta_0 g \left[AO(1) + O(1) \right] + O(\mathcal{H}_\phi).
\end{aligned}
\tag{3.16}
$$

Also, with the similar computations (2.23) in the Lemma 2.1,

$$
\begin{aligned}
& \sum_{\alpha\beta=1}^{n} F^{\alpha\beta}\frac{\widetilde{a}_{ij,\alpha}\widetilde{a}_{ij,\beta}}{\widetilde{a}_{ii}} - \frac{1}{u_n^3}[2u_n^2 u_{nj}\sum_{\alpha=1}^{n} F^{\alpha i}a_{ij,\alpha} + u_{nj}^2 F^{ii}u_{ii}] \\
= \;& \sum_{\alpha\beta=1}^{n} F^{\alpha\beta}\frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}} + \eta_0 g\sum_{\alpha\beta=1}^{n} F^{\alpha\beta}\frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}\widetilde{a}_{ii}} \\
& -\frac{1}{u_n^3}[2u_n^2 u_{nj}\sum_{\alpha=1}^{n} F^{\alpha i}a_{ij,\alpha} + u_{nj}^2 F^{ii}u_{ii}] \\
= \;& \eta_0 g\sum_{\alpha\beta=1}^{n} F^{\alpha\beta}\frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}\widetilde{a}_{ii}} \\
& -\frac{1}{u_n^3}\sum_{\alpha\beta=1}^{n} F^{\alpha\beta}\frac{1}{u_{ii}}[-u_n u_{ij\alpha} + u_{nj}u_{i\alpha} + u_{ni}u_{j\alpha}][-u_n u_{ij\beta} + u_{nj}u_{i\beta} + u_{ni}u_{j\beta}] \\
& -\frac{1}{u_n^3}[2u_{nj}\sum_{\alpha=1}^{n} F^{\alpha i}(-u_n u_{ij\alpha} + u_{nj}u_{i\alpha} + u_{ni}u_{j\alpha}) + u_{nj}^2 F^{ii}u_{ii}] \\
= \;& \eta_0 g\sum_{\alpha\beta=1}^{n} F^{\alpha\beta}\frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}\widetilde{a}_{ii}} \\
& -\frac{1}{u_n^3}\sum_{\alpha\beta=1}^{n} F^{\alpha\beta}\frac{1}{u_{ii}}[-u_n u_{ij\alpha} + 2u_{nj}u_{i\alpha}][-u_n u_{ij\beta} + 2u_{nj}u_{i\beta}] \\
& -\frac{1}{u_n^3}u_{jj}[\sum_{\alpha=1}^{n-1} F^{\alpha j}\frac{2}{u_{ii}}u_{ni}(-u_n u_{ij\alpha} + u_{nj}u_{i\alpha}) + F^{ii}\frac{1}{u_{ii}}u_{jj}u_{ni}^2 \\
& \qquad +2F^{jn}\frac{1}{u_{ii}}u_{ni}(-u_n u_{ijn} + 2u_{nj}u_{in}) + F^{ij}u_{ni}u_{nj}] \\
= \;& -\frac{1}{u_n^3}\sum_{\alpha\beta=1}^{n} F^{\alpha\beta}\frac{1}{u_{ii}}[-u_n u_{ij\alpha} + 2u_{nj}u_{i\alpha}][-u_n u_{ij\beta} + 2u_{nj}u_{i\beta}] \\
& +\eta_0 g\left[\sum_{\alpha\beta=1}^{n} F^{\alpha\beta}\frac{a_{ij,\alpha}a_{ij,\beta}}{a_{ii}\widetilde{a}_{ii}} + O(1).\right]
\end{aligned}
\tag{3.17}
$$

From the above calculations, the proof is complete. □

Theorem 1.2 is a direct consequence of the following proposition and the strong maximum principle.

Proposition 3.2. Suppose that the function F, u satisfy assumptions in Theorem 1.2. If the second fundamental form b_{ij} of $\Sigma^{u(x,t_0)}$ attains minimum rank $l = l(t_0)$ at certain point $x_0 \in \Omega$, then there exist a neighborhood $\mathcal{O}\times(t_0-\delta_0, t_0+\delta_0]$ of (x_0, t_0) and a positive

constant C independent of ϕ (defined in (3.5)), such that

$$\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\phi_{\alpha\beta}(x,t) - \phi_t \leqslant C(\phi + |\nabla\phi|) + \eta_0 g\left[-A^2F^{nn}u_n^2 + AO(1) + O(1)\right] \tag{3.18}$$

holds for any $(x,t) \in \mathcal{O} \times (t_0 - \delta_0, t_0 + \delta_0]$.

Proof: Since

$$u_t = F(\nabla^2 u, \nabla u, u, t), \tag{3.19}$$

for each $j \in B$, differentiating the above equation in e_j direction at x,

$$u_{jt} = \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta j} + F^{u_n}u_{jn} + O(\mathcal{H}_\phi) \tag{3.20}$$

and

$$\begin{aligned} u_{jjt} \quad = \quad & \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta jj} + \sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}u_{\alpha\beta j}u_{rsj} + 2\sum_{\alpha,\beta,l=1}^{n} F^{\alpha\beta,u_l}u_{\alpha\beta j}u_{lj} \\ & +2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,u}u_{j\alpha\beta}u_j + \sum_{l,s=1}^{n} F^{u_l,u_s}u_{lj}u_{sj} - 2\sum_{l=1}^{n} F^{u_l,u}u_{lj}u_j \\ & +F^{u,u}u_j^2 + \sum_{l=1}^{n} F^{u_l}u_{ljj} + F^u u_{jj}. \end{aligned} \tag{3.21}$$

It follows from (3.9)and (3.13) that, at (x,t)

$$\sum_{\alpha\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta j} - u_{jt} = -F^{p_n}u_{nj} + \eta_0 g F^{p_j}u_n + O(\mathcal{H}_\phi) \tag{3.22}$$

and

$$\begin{aligned} & \sum_{\alpha\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta jj} - u_{jjt} \\ = \quad & -\sum_{\alpha\beta\gamma\eta=1}^{n} F^{\alpha\beta,\gamma\eta}u_{\alpha\beta j}u_{\gamma\eta j} - 2\sum_{\alpha\beta=1}^{n} F^{\alpha\beta,p_n}u_{\alpha\beta j}u_{nj} - F^{p_n,p_n}u_{nj}u_{nj} - 2\frac{F^{p_n}}{u_n}u_{nj}^2 \\ & +\eta_0 g[-AF^{p_n}u_n^2] \\ & +\eta_0 g[2\sum_{\alpha\beta=1}^{n} F^{\alpha\beta,p_j}u_{\alpha\beta j}u_n + F^{p_j,p_j}u_{jj}u_n + 2F^{p_n,p_j}u_{nj}u_n + F^{p_n}u_n + 2F^{p_j}u_{jn} + F^{p_l}u_{nl}] \\ & +O(\mathcal{H}_\phi). \end{aligned}$$

From lemma 3.1,

$$
\begin{aligned}
& F^{\alpha\beta}\phi_{\alpha\beta} - \phi_t \\
= & \sum_{j\in B} u_n^{-3}\left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right]\Bigg\{\Big[\sum_{\alpha,\beta,r,s=1}^{n} F^{\alpha\beta,rs}u_{\alpha\beta j}u_{rsj} \\
& \qquad + 2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta,u_n}u_{j\alpha\beta}u_{jn} + F^{u_n,u_n}u_{jn}^2\Big]u_n^2 \\
& \qquad + 4u_{jn}u_n\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{j\alpha\beta} - 6u_{jn}^2\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}u_{\alpha\beta}\Bigg\} \\
& + 2u_n^{-3}\sum_{j\in B, i\in G}\left[\sigma_l(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)}\right]\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\frac{1}{u_{ii}}[u_n u_{ij\alpha} - 2u_{i\alpha}u_{jn}][u_n u_{ij\beta} - 2u_{i\beta}u_{jn}] \\
& + \eta_0 g\left[-A^2F^{nn}u_n^2 + AO(1) + O(1)\right] \\
& - \frac{1}{\sigma_1^3(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\in B} F^{\alpha\beta}[\sigma_1(B)\widetilde{a}_{ii,\alpha} - \widetilde{a}_{ii}\sum_{j\in B}\widetilde{a}_{jj,\alpha}][\sigma_1(B)\widetilde{a}_{ii,\beta} - \widetilde{a}_{ii}\sum_{j\in B}\widetilde{a}_{jj,\beta}] \\
& - \frac{1}{\sigma_1(B)}\sum_{\alpha,\beta=1}^{n}\sum_{i\neq j\in B} F^{\alpha\beta}\widetilde{a}_{ij,\alpha}\widetilde{a}_{ij,\beta} + O(\mathcal{H}_\phi).
\end{aligned}
$$

So, following the argument in the proof of Proposition 2.3, we get,

$$
\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\phi_{\alpha\beta}(x,t) - \phi_t \leqslant C(\phi + |\nabla\phi|) + \eta_0 g\left[-A^2F^{nn}u_n^2 + AO(1) + O(1)\right]. \tag{3.23}
$$

The proof is completed. $\square$

References

1. Ahlfors, L.V.: *Conformal invariants: topics in geometric function theory.* McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York-Dsseldorf-Johannesburg(1973)
2. Bian, B., Guan, P.: A microscopic convexity principle for nonlinear partial differential equations. Inventiones Math. **177**, 307-335(2009)
3. Bian, B., Guan, P., Ma, X.N., Xu, L.: A constant rank theorem for quasiconcave solutions of fully nonlinear partial differential equations. to appear in Indiana Univ. Math. J..
4. Bianchini, C., Longinetti, M., Salani, P.: Quasiconcave solutions to elliptic problems in convex rings, Indiana Univ. Math. J. **58**, 1565-1590(2009)
5. Borell, C.: Brownian motion in a convex ring and quasi-concavity. Commun. Math. Phys. **86**, 143-147(1982)
6. Caffarelli, L., Friedman, A.: Convexity of solutions of some semilinear elliptic equations. Duke Math. J. **52**, 431–455(1985)
7. Caffarelli, L., Spruck, J.: Convexity properties of solutions to some classical variational problems. Comm. Part. Diff. Eq. **7**, 1337-1379(1982)
8. Chang, S.-Y.A., Ma, X.N., Yang, P.: Principal curvature estimates for the convex level sets of semilinear elliptic equations. Discrete Contin. Dyn. Syst. **28**, 1151–1164(2010)

9. Gabriel, R.: A result concerning convex level surfaces of 3-dimensional harmonic functions. J. London Math. Soc. **32**, 286-294(1957)
10. Guan, P., Ma, X.N.: The Christoffel-Minkowski problem I: convexity of solutions of a Hessian equations. Inventiones Math. **151**, 553-577(2003)
11. Guan, P., Xu, L.: Convexity estimates for level surfaces of quasiconcave solutions to fully nonlinear elliptic equations. http://arxiv.org/abs/1004.1187v1
12. Kawhol, B.: *Rearrangements and convexity of level sets in PDE*. Springer Lecture Notes in Math.**1150**(1985)
13. Korevaar, N.: Convexity of level sets for solutions to elliptic ring problems. Comm. Part. Diff. Eq. **15**(4), 541-556(1990)
14. Lewis, J.: Capacitary functions in convex rings. Arch. Rat. Mech. Anal. **66**, 201-224(1977)
15. Longinetti, M.: Convexity of the level lines of harmonic functions. (Italian) Boll. Un. Mat. Ital. A **6**, 71–75(1983)
16. Longinetti, M.: On minimal surfaces bounded by two convex curves in parallel planes. J. Diff. Equations **67**, 344–358(1987)
17. Ma, X.N., Ou, Q.Z., Zhang, W.: Gaussian curvature estimates for the convex level sets of p-harmonic functions. Comm. Pure Appl. Math.**63**, 0935–0971(2010)
18. Ma, X.N., Zhang, W.: The concavity of the Gaussian curvature of the convex level sets of p-harmonic functions with respect to the height. Preprint
19. Ortel, M., Schneider, W.: Curvature of level curves of harmonic functions. Canad. Math. Bull. **26**, 399–405(1983)
20. Shiffman, M.: On surfaces of stationary area bounded by two circles or convex curves in parallel planes. Annals of Math. **63**, 77–90(1956)
21. Talenti, G.: On functions whose lines of steepest descent bend proportionally to level lines. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **10**(4), 587–605(1983)
22. Wang, P.H., Zhang, W.: Gaussian curvature estimates for the convex level sets for some nonlinear partial differential equations. http://arxiv.org/abs/1003.2057v1
23. Xu, L.: A microscopic convexity theorem of level sets for solutions to elliptic equations. Cal. Var. PDE. DOI 10.1007/s00526-010-0333-3(2010)

Department of Mathematics, University of Science and Technology of China, Hefei 230026, Anhui Province, CHINA.

E-mail address: `cqchen@mail.ustc.edu.cn`

Department of Mathematics, University of Science and Technology of China, Hefei 230026, Anhui Province, CHINA, and School of Mathematical Sciences, Harbin Normal University, Harbin 150025, Heilongjiang Province, CHINA.

E-mail address: `shjshi@mail.ustc.edu.cn`